

ON CLASSES OF C3 AND D3 MODULES

ABYZOV ADEL NAILEVICH, TRUONG CONG QUYNH
AND TRAN HOAI NGOC NHAN

ABSTRACT. The aim of this paper is to study the notions of \mathcal{A} -C3 and \mathcal{A} -D3 modules for some class \mathcal{A} of right modules. Several characterizations of these modules are provided and used to describe some well-known classes of rings and modules. For example, a regular right R -module F is a V -module if and only if every F -cyclic module M is an \mathcal{A} -C3 module where \mathcal{A} is the class of all simple submodules of M . Moreover, let R be a right artinian ring and \mathcal{A} , a class of right R -modules with local endomorphisms, containing all simple right R -modules and closed under isomorphisms. If all right R -modules are \mathcal{A} -injective, then R is a serial artinian ring with $J^2(R) = 0$ if and only if every \mathcal{A} -C3 right R -module is quasi-injective, if and only if every \mathcal{A} -C3 right R -module is C3.

1. INTRODUCTION AND NOTATION.

The study of modules with summand intersection property was motivated by the following result of Kaplansky: every free module over a commutative principal ideal ring has the summand intersection property (see [14, Exercise 51(b)]). A module M is said to have the *summand intersection property* if the intersection of any two direct summands of M is a direct summand of M . This definition is introduced by Wilson [18]. Dually, Garcia [10] consider the summand sum property. A module M is said to have the *summand sum property* if the sum of any two direct summands is a direct summand of M . These properties have been studied by several authors (see [1, 3, 11, 12, 17],...). Moreover, the classes of C3-modules and D3-modules have recently studied by Yousif et al. in [4, 20]. Some characterizations of semisimple rings and regular rings and other classes of rings are studied via C3-modules and D3-modules. On the other hand, several authors investigated some properties of generalizations of C3-modules and D3-modules in [6, 13]; namely, simple-direct-injective modules and simple-direct-projective modules. A right R -module M is called a *C3-module* if, whenever A and B are submodules of M with $A \subset_d M$, $B \subset_d M$ and $A \cap B = 0$, then $A \oplus B \subset_d M$. M is called *simple-direct-injective* in [6] if the submodules A and B in the above definition are simple. Dually, M is called a *D3-module* if, whenever M_1 and M_2 are direct summands of M

2010 *Mathematics Subject Classification.* 16D40, 16D80.

Key words and phrases. \mathcal{A} -C3 module, \mathcal{A} -D3 module, V -module.

and $M = M_1 + M_2$, then $M_1 \cap M_2$ is a direct summand of M . M is called *simple-direct-projective* in [13] if the submodules M_1 and M_2 in the above definition are maximal.

In Section 2, we introduce the notions of \mathcal{A} -C3 modules and \mathcal{A} -D3 modules, where \mathcal{A} is a class of right modules over the ring R and closed under isomorphisms. It is shown that if each factor module of M is \mathcal{A} -injective, then M is an \mathcal{A} -D3 module if and only if M satisfies D2 for the class \mathcal{A} , if and only if M have the summand intersection property for the class \mathcal{A} in Proposition 2.7. On the other hand, if every submodule of M is \mathcal{A} -projective, then M is an \mathcal{A} -C3 module if and only if M satisfies C2 for the class \mathcal{A} , if and only if M have the summand sum property for the class \mathcal{A} in Proposition 2.13. Some well-known properties of other modules are obtained from these results.

In Section 3, we provide some characterizations of serial artinian rings and semisimple artinian rings. The Theorem 3.2 and Theorem 3.3 are indicated that let R be a right artinian ring and \mathcal{A} , a class of right R -modules with local endomorphisms, containing all simple right R -modules and closed under isomorphisms:

- (1) If all right R -modules are \mathcal{A} -injective, the following conditions are equivalent for a ring R :
 - (i) R is a serial artinian ring with $J^2(R) = 0$.
 - (ii) Every \mathcal{A} -C3 right R -module is quasi-injective.
 - (iii) Every \mathcal{A} -C3 right R -module is C3.
- (2) If all right R -modules are \mathcal{A} -projective, then the following conditions are equivalent for a ring R :
 - (i) R is a serial artinian ring with $J^2(R) = 0$.
 - (ii) Every \mathcal{A} -D3 right R -module is quasi-projective.
 - (iii) Every \mathcal{A} -D3 right R -module is D3.

Moreover, we give an equivalent condition for a regular V -module. It is shown that a regular right R -module F is a V -module if and only if every F -cyclic module is simple-direct-injective in Theorem 3.9. It is an extension the result of rings to modules.

Throughout this paper R denotes an associative ring with identity, and modules will be unitary right R -modules. The Jacobson radical ideal in R is denoted by $J(R)$. The notations $N \leq M$, $N \leq_e M$, $N \trianglelefteq M$, or $N \subset_d M$ mean that N is a submodule, an essential submodule, a fully invariant submodule, and a direct summand of M , respectively. Let M and N be right R -modules. M is called N -injective if for any right R -module K and any monomorphism $f : K \rightarrow N$, the induced homomorphism $\text{Hom}(N, M) \rightarrow \text{Hom}(K, M)$ by f is an epimorphism. M is called N -projective if for any right R -module K and any epimorphism $f : N \rightarrow K$, the induced homomorphism $\text{Hom}(M, N) \rightarrow \text{Hom}(M, K)$ by f is an epimorphism. Let \mathcal{A} be a class of right modules over the ring R . M is called \mathcal{A} -injective (\mathcal{A} -projective) if M is N -injective (resp., N -projective) for all $N \in \mathcal{A}$. We refer to [5], [7], [16], and [19] for all the undefined notions in this paper.

2. ON \mathcal{A} -C3 MODULES AND \mathcal{A} -D3 MODULES

Let \mathcal{A} be a class of right modules over a ring R and closed under isomorphisms. We call that a right R -module M is an \mathcal{A} -C3 module if, whenever $A \in \mathcal{A}$ and $B \in \mathcal{A}$ are submodules of M with $A \subset_d M$, $B \subset_d M$ and $A \cap B = 0$, then $A \oplus B \subset_d M$. Dually, M is an \mathcal{A} -D3 module if, whenever M_1 and M_2 are direct summands of M with $M/M_1, M/M_2 \in \mathcal{A}$ and $M = M_1 + M_2$, then $M_1 \cap M_2$ is a direct summand of M .

Remark 2.1. Let M be a right R -module and \mathcal{A} , a class of right R -modules.

- (1) If M is a C3 (D3) module, then M is an \mathcal{A} -C3 (resp., \mathcal{A} -D3) module.
- (2) If $\mathcal{A} = \text{Mod} - R$, then \mathcal{A} -C3 modules (\mathcal{A} -D3 modules) modules are precisely the C3 modules (resp., D3) modules.
- (3) If \mathcal{A} is the class of all simple submodules of M , then \mathcal{A} -C3 (\mathcal{A} -D3) modules are precisely the simple-direct-injective (resp., simple-direct-projective) modules and studied in [6, 13].
- (4) If \mathcal{A} is a class of injective right R -modules, then M is always an \mathcal{A} -C3 module.
- (5) If \mathcal{A} is a class of projective right R -modules, then M is always an \mathcal{A} -D3 module.

Lemma 2.2. *Let \mathcal{A} be a class of right R -modules and closed under isomorphisms. Then every summand of an \mathcal{A} -C3 module (\mathcal{A} -D3 module) is also an \mathcal{A} -C3 module (resp., \mathcal{A} -D3 module).*

Proof. The proof is straightforward. □

Proposition 2.3. *Let \mathcal{A} be a class of right R -modules and closed under direct summands. Then the following conditions are equivalent for a module M :*

- (1) M is an \mathcal{A} -C3 module.
- (2) If $A \in \mathcal{A}$ and $B \in \mathcal{A}$ are submodules of M with $A \subset_d M$, $B \subset_d M$ and $A \cap B = 0$, there exist submodules A_1 and B_1 of M such that $M = A \oplus B_1 = A_1 \oplus B$ with $A \leq A_1$ and $B \leq B_1$.
- (3) If $A \in \mathcal{A}$ and $B \in \mathcal{A}$ are submodules of M with $A \subset_d M$, $B \subset_d M$ and $A \cap B \subset_d M$, then $A + B \subset_d M$.

Proof. It is similar to the proof of Proposition 2.2 in [4]. □

Dually Proposition 2.3, we have the following proposition.

Proposition 2.4. *Let \mathcal{A} be a class of right R -modules and closed under isomorphisms. Then the following conditions are equivalent for a module M :*

- (1) M is an \mathcal{A} -D3 module.
- (2) If $M/A, M/B \in \mathcal{A}$ with $A \subset_d M$, $B \subset_d M$ and $M = A + B$, then $M = A \oplus B_1 = A_1 \oplus B$ with $A_1 \leq A$ and $B_1 \leq B$.
- (3) If $M/A, M/B \in \mathcal{A}$ with $A \subset_d M$, $B \subset_d M$ and $A + B \subset_d M$, then $A \cap B \subset_d M$.

Let $f : A \rightarrow B$ be a homomorphism. We denote by $\langle f \rangle$ the submodule of $A \oplus B$ as follows:

$$\langle f \rangle = \{a + f(a) \mid a \in A\}.$$

The following result is proved in Lemma 2.6 of [15].

Lemma 2.5. *Let $M = X \oplus Y$ and $f : A \rightarrow Y$, a homomorphism with $A \leq X$. Then the following conditions hold*

- (1) $A \oplus Y = \langle f \rangle \oplus Y$.
- (2) $\text{Ker}(f) = X \cap \langle f \rangle$.

Proposition 2.6. *Let M be an \mathcal{A} -D3 module with \mathcal{A} a class of right R -modules and closed under isomorphisms and summands. If $M = M_1 \oplus M_2$ and $f : M_1 \rightarrow M_2$ is a homomorphism with $\text{Im}(f) \subset_d M_2$ and $\text{Im}(f) \in \mathcal{A}$, then $\text{Ker}(f)$ is a direct summand of M_1 .*

Proof. Assume that $M = M_1 \oplus M_2$ and a homomorphism $f : M_1 \rightarrow M_2$ with $\text{Im}(f) \subset_d M_2$ and $\text{Im}(f) \in \mathcal{A}$. Call $M' := M_1 \oplus \text{Im}(f)$. Then M' is a direct summand of M and so is an \mathcal{A} -D3 module. It follows that $M' = M_1 \oplus \text{Im}(f) = \langle f \rangle \oplus \text{Im}(f)$ by Lemma 2.5. It is easily to check $M'/M_1, M'/\langle f \rangle \in \mathcal{A}$ and $M' = M_1 + \langle f \rangle$. As M' is an \mathcal{A} -D3 module and by Lemma 2.5, $\langle f \rangle \cap M_1 = \text{Ker}(f)$ is a direct summand of M' . Thus $\text{Ker}(f)$ is a direct summand of M_1 . \square

Proposition 2.7. *Let M be a right R -module and \mathcal{A} , a class of right R -modules and closed under isomorphisms and summands. If each factor module of M is \mathcal{A} -injective, then the following conditions are equivalent:*

- (1) *For any two direct summands M_1, M_2 of M such that $M/M_1, M/M_2 \in \mathcal{A}$, $M_1 \cap M_2$ is a direct summand of M .*
- (2) *M is an \mathcal{A} -D3 module.*
- (3) *Any submodule N of M such that the factor module $M/N \in \mathcal{A}$ is isomorphic to a direct summand of M , is a direct summand of M .*
- (4) *For any decomposition $M = M_1 \oplus M_2$ with $M_2 \in \mathcal{A}$, then every homomorphism $f : M_1 \rightarrow M_2$ has the kernel a direct summand of M_1 .*
- (5) *Whenever X_1, \dots, X_n are direct summands of M and $M/X_1, \dots, M/X_n \in \mathcal{A}$, then $\cap_{i=1}^n X_i$ is a direct summand of M .*

Proof. (2) \Rightarrow (1). Let M_1, M_2 be direct summands of M such that $M/M_1, M/M_2 \in \mathcal{A}$. Then $M = M_1 \oplus M'_1$. Without loss of generality we can assume that $M_2 \not\subseteq M_1, M_2 \not\subseteq M'_1$. From our assumption, $\pi(M_2)$ is a direct summand of M'_1 . Then we can write $M'_1 = \pi(M_2) \oplus M''_1$ for some $M''_1 \leq M'_1$. Since the class \mathcal{A} is closed under direct summands, $M''_1 \in \mathcal{A}$. It is easy to see that $M_1 + M''_1$ is a direct summand of M . We have $M/(M_1 + M''_1) \in \mathcal{A}$ and $M_1 + M''_1 + M_2 = M$. It follows that $M_1 \cap M_2 = (M_1 + M''_1) \cap M_2$ is a direct summand of M .

(3) \Rightarrow (2). It is obvious.

(1) \Rightarrow (4). Assume that $M = M_1 \oplus M_2$ with $M_2 \in \mathcal{A}$ and a homomorphism $f : M_1 \rightarrow M_2$. It follows that $M = M_1 \oplus M_2 = \langle f \rangle \oplus M_2$ by Lemma 2.5. Note that $M/M_1, M/\langle f \rangle \in \mathcal{A}$. By (1) and Lemma 2.5, $\langle f \rangle \cap M_1 = \text{Ker}(f)$ is a direct summand of M . Thus $\text{Ker}(f)$ is a direct summand of M_1 .

(4) \Rightarrow (3). Let M_1, M_2 be submodules of M such that $M = M_1 \oplus A$, $M/M_2 \cong A$ and $A \in \mathcal{A}$. Call $\pi_1 : M \rightarrow M_1$ and $\pi_2 : M \rightarrow A$ the projections. By the hypothesis, $\pi_2(M_2)$ is a direct summand of A and hence $A = \pi_2(M_2) \oplus B$ for some submodule B of A . Call $p : M \rightarrow M/M_2$ the canonical projection and isomorphism $\phi : M/M_2 \rightarrow A$. Take the homomorphism $f = \phi \circ (p|_{M_1}) : M_1 \rightarrow A$. It follows that $\text{Ker}(f) = M_1 \cap M_2$. By (4), $\text{Ker}(f) = M_1 \cap M_2$ is a direct summand of M_1 . Call N_1 a submodule of M_1 with $M_1 = N_1 \oplus (M_1 \cap M_2)$. Note that $M_1 + M_2 = M_1 \oplus \pi_2(M_2)$ and $N_1 \cap M_2 = 0$. This gives that

$$\begin{aligned} M &= M_1 \oplus \pi_2(M_2) \oplus B \\ &= (M_1 + M_2) \oplus B \\ &= [N_1 \oplus (M_1 \cap M_2) + M_2] \oplus B = (N_1 + M_2) \oplus B \\ &= (N_1 \oplus M_2) \oplus B. \end{aligned}$$

(1) \Rightarrow (5). We prove this by induction on n . When $n = 2$, the assertion is true from (1). Suppose that the assertion is true for $n = k$. Let X_1, X_2, \dots, X_{k+1} be summands of M and $M/X_1, M/X_2, \dots, M/X_{k+1} \in \mathcal{A}$. We can write $M = \cap_{i=1}^k X_i \oplus N$ for some submodule N of M . Without loss of generality we can assume that $\cap_{i=1}^k X_i \not\subseteq X_{k+1}$. Let $f : M \rightarrow M/X_{k+1}$ be the natural projection. Then $(\cap_{i=1}^k X_i)/[(\cap_{i=1}^k X_i) \cap X_{k+1}]$ is \mathcal{A} -injective, and therefore, it is isomorphic to a direct summand of $M/X_{k+1} \in \mathcal{A}$. This gives that $\cap_{i=1}^k X_i / \cap_{i=1}^{k+1} X_i$ is isomorphic to a direct summand of M and

$$M/(\cap_{i=1}^{k+1} X_i \oplus N) = (\cap_{i=1}^k X_i \oplus N)/(\cap_{i=1}^{k+1} X_i \oplus N) \in \mathcal{A}.$$

Since the equivalence of (1) and (3), $(\cap_{i=1}^{k+1} X_i) \oplus N$ is a direct summand of M . Thus

$\bigcap_{i=1}^{k+1} X_i$ is a direct summand of M . □

Corollary 2.8. *The following conditions are equivalent for a module M :*

- (1) *If M/A is a semisimple module and B , a submodule of M with $M/A \cong B \subset_d M$, then $A \subset_d M$.*
- (2) *For any two direct summands A, B of M with M/A and M/B are semisimple modules, then $A \cap B \subset_d M$.*
- (3) *For any two direct summands A, B of M such that $M/A, M/B$ are semisimple modules and $A + B = M$, then $A \cap B$ is a direct summand of M .*
- (4) *Whenever X_1, X_2, \dots, X_n are direct summands of M and $M/X_1, M/X_2, \dots, M/X_n$ are semisimple modules, then $\cap_{i=1}^n X_i$ is a direct summand of M .*

Corollary 2.9. *Let P be a quasi-projective module. If X_1, \dots, X_n are summands of P and $P/X_1, \dots, P/X_n$ are semisimple modules, then $\cap_{i=1}^n X_i$ is a direct summand of P .*

Corollary 2.10. *The following conditions are equivalent for a module M :*

- (1) *For any maximal submodule A of M and any submodule B of M such that $M/A \cong B \subset_d M$, $A \subset_d M$.*
- (2) *For any two maximal summands A, B of M , $A \cap B \subset_d M$.*
- (3) *If M/A is a finitely generated semisimple module with $M/A \cong B \subset_d M$, then $A \subset_d M$.*
- (4) *Whenever X_1, X_2, \dots, X_n are maximal summands of M , then $\cap_{i=1}^n X_i$ is a direct summand of M .*

Proof. (1) \Leftrightarrow (2) \Leftrightarrow (4). Follow from Proposition 2.7.

(3) \Rightarrow (1). Clearly.

(1) \Rightarrow (3). Assume that M/A is a finitely generated semisimple module and isomorphic to a direct summand of M . Write $M/A = M_1/A \oplus \dots \oplus M_n/A$ with simple submodules M_i/A of M/A . Then $M_i \cap (\sum_{j \neq i} M_j) = A$ for all $i = 1, 2, \dots, n$. For any subset $\{i_1, i_2, \dots, i_{n-1}\}$ of the set $I := \{1, 2, \dots, n\}$, it is easily to see that

$$M/(M_{i_1} + M_{i_2} + \dots + M_{i_{n-1}}) \simeq M_k/A$$

for some $k \in I \setminus \{i_1, i_2, \dots, i_{n-1}\}$. It follows that $M/(M_{i_1} + M_{i_2} + \dots + M_{i_{n-1}})$ is isomorphic to a simple summand of M . By (1), $M_{i_1} + M_{i_2} + \dots + M_{i_{n-1}}$ is a maximal summand of M . On the other hand, we can check that

$$A = \bigcap_{\{i_1, i_2, \dots, i_{n-1}\} \subset I} (M_{i_1} + M_{i_2} + \dots + M_{i_{n-1}}).$$

So, by (4), A is a direct summand of M . \square

Proposition 2.11. *Let M be an \mathcal{A} -C3 module with \mathcal{A} a class of right R -modules and closed under isomorphisms and summands. If $M = A_1 \oplus A_2$ and $f : A_1 \rightarrow A_2$ is a homomorphism with $\text{Ker}(f) \in \mathcal{A}$ and $\text{Ker}(f) \subset_d A_1$, then $\text{Im}(f)$ a direct summand of A_2 .*

Proof. Let $f : A_1 \rightarrow A_2$ be an R -homomorphism with $\text{Ker}(f) \in \mathcal{A}$. By the hypothesis, there exists a decomposition $A_1 = \text{Ker}(f) \oplus B$ for a submodule B of A_1 . Then $B \oplus A_2$ is a direct summand of M . Note that every direct summand of an \mathcal{A} -C3 module is also an \mathcal{A} -C3 module. Hence $B \oplus A_2$ is an \mathcal{A} -C3 module. Let $g = f|_B : B \rightarrow A_2$. Then g is a monomorphism and $\text{Im}(g) = \text{Im}(f)$. It is easy to see that $B \oplus A_2 = \langle g \rangle \oplus A_2$, $\langle g \rangle \cap B = 0$ and $\langle g \rangle \simeq B$. Note that $B, \langle g \rangle \in \mathcal{A}$. As $B \oplus A_2$ is an \mathcal{A} -C3 module, $B \oplus \langle g \rangle$ is a direct summand of $B \oplus A_2$. Thus $B \oplus \langle g \rangle = B \oplus \text{Im}(g)$, which implies that $\text{Im}(g)$ or $\text{Im}(f)$ is a direct summand of A_2 . \square

Proposition 2.12. *Let M be a right R -module and \mathcal{A} , a class of right R -modules and closed under isomorphisms and summands. If every submodule of M is \mathcal{A} -projective, the following conditions are equivalent:*

- (1) *For any two direct summands M_1, M_2 of M such that $M_1, M_2 \in \mathcal{A}$, $M_1 + M_2$ is a direct summand of M .*

- (2) M is an \mathcal{A} -C3 module.
 (3) For any decomposition $M = A_1 \oplus A_2$ with $A_1 \in \mathcal{A}$, then every homomorphism $f : A_1 \rightarrow A_2$ has the image a direct summand of A_2 .

Proof. (1) \Rightarrow (2) is obvious.

(2) \Rightarrow (3) Let $f : A_1 \rightarrow A_2$ be an R -homomorphism with $A_1 \in \mathcal{A}$. By the hypothesis, $\text{Ker}(f)$ is a direct summand of A_1 . The rest of proof is followed from Proposition 2.11.

(3) \Rightarrow (1) Let N and K be direct summands of M such that $N, K \in \mathcal{A}$. Write $M = N \oplus N'$ and $M = K \oplus K'$ for some submodules N', K' of M . Consider the canonical projections $\pi_K : M \rightarrow K$ and $\pi_{N'} : M \rightarrow N'$. Let $A = \pi_{N'}(\pi_K(N))$. Then $A = (N + K) \cap (N + K') \cap N'$ is a direct summand of M by (3). Write $M = A \oplus L$ for some submodule L of M . Clearly,

$$(N + K) \cap [(N + K') \cap (N' \cap L)] = 0.$$

Hence, $N' = A \oplus (N' \cap L)$ and $M = (N \oplus A) \oplus (N' \cap L)$. Since $A \leq N + K$ and $A \leq N + K'$, we get

$$N + K = (N \oplus A) \cap [(N + K) \cap (N' \cap L)]$$

and

$$N + K' = (N \oplus A) \cap [(N + K') \cap (N' \cap L)].$$

They imply

$$\begin{aligned} M &= N + K' + K \\ &= (N \oplus A) + [(N + K) \cap (N' \cap L)] + [(N + K') \cap (N' \cap L)] \\ &\leq (N + K) + [(N + K') \cap (N' \cap L)]. \end{aligned}$$

Thus $M = (N + K) \oplus [(N + K') \cap (N' \cap L)]$. \square

Proposition 2.13. *Let M be a right R -module and \mathcal{A} , a class of artinian right R -modules and closed under isomorphisms and summands. If every submodule of M is \mathcal{A} -projective, then the following conditions are equivalent:*

- (1) M is an \mathcal{A} -C3 module.
 (2) Every submodule $N \in \mathcal{A}$ of M that is isomorphic to a direct summand of M is itself a direct summand.
 (3) Whenever X_1, X_2, \dots, X_n are direct summands of M and $X_1, X_2, \dots, X_n \in \mathcal{A}$, then $\sum_{i=1}^n X_i$ is a direct summand of M .

Proof. (1) \Rightarrow (2). Let M_1 be submodule of M and isomorphic to a direct summand M_2 of M and $M_1 \in \mathcal{A}$. Then $M = M_2 \oplus M'_2$. If $M_1 \subset M_2$, then by M_2 is artinian and $M_1 \cong M_2$, implies that $M_1 = M_2$. Let $M_1 \not\subset M_2$ and $\pi : M_2 \oplus M'_2 \rightarrow M'_2$ be projection. According to the hypothesis, $\text{Ker}(\pi|_{M_1})$ is a direct summand of M_1 . It follows that $M_1 = M_1 \cap M_2 \oplus N_1$. Since $N_1 \cong \pi(M_1)$, $M_1 \cong M_2$, then there is an isomorphism $\phi : N' \rightarrow \pi(M_1)$, where N' is a direct summand of M_1 . Since $\langle \phi \rangle \in \mathcal{A}$ and $\langle \phi \rangle \cap M_2 = 0$, $M_2 + \langle \phi \rangle = M_2 \oplus N_1$ is a direct summand of M . Therefore, N_1 is a non-zero direct

summand of M . It is clear that $M_1 \cap M_2 \in \mathcal{A}$ and $M_1 \cap M_2$ is isomorphic to a direct summand of M . If $M_1 \cap M_2$ is not a direct summand of M , by using a argument that are similar to the argument presented above, we can show that $M_1 \cap M_2 = N_2 \oplus N'_2$, where $N_2 \in \mathcal{A}$ is a non-zero direct summand of M and $N'_2 \in \mathcal{A}$ is a submodule of M isomorphic to a direct summand of M . Since each module of the class \mathcal{A} is artinian, by conducting similar constructions continue for some k , we obtain a decomposition $M_1 = N_1 \oplus \dots \oplus N_k$, where N_i is a direct summand of M and $N_i \in \mathcal{A}$ for each i . Since M is an \mathcal{A} -C3 module, $N_1 \oplus N_2 \oplus \dots \oplus N_k$ is a direct summand of M .

(2) \Rightarrow (1). It is obvious.

(1) \Rightarrow (3). We prove this by induction on n . When $n = 2$, the assertion follows from Proposition 2.12. Suppose that the assertion is true for $n = k$. Let X_1, X_2, \dots, X_{k+1} be summands of M and $X_1, X_2, \dots, X_{k+1} \in \mathcal{A}$. Then there exists a submodule N of M such that $M = (\sum_{i=1}^k X_i) \oplus N$. Let $\pi : (\sum_{i=1}^k X_i) \oplus N \rightarrow N$ be the natural projection. As $\pi(X_{k+1})$ is \mathcal{A} -projective, then $X_{k+1} = ((\sum_{i=1}^k X_i) \cap X_{k+1}) \oplus S$ for some submodule S of M . Since the equivalence of (1) and (2), $\pi(X_{k+1})$ is a direct summand of M and, therefore, $N = \pi(X_{k+1}) \oplus T$ with T a submodule M . It follows that $\sum_{i=1}^{k+1} X_i = (\sum_{i=1}^k X_i) \oplus \pi(X_{k+1})$ and $M = (\sum_{i=1}^k X_i) \oplus \pi(X_{k+1}) \oplus T$. Thus, $\sum_{i=1}^{k+1} X_i$ is a direct summand of M . \square

Remark 2.14. Let F be any nonzero free module over \mathbb{Z} and \mathcal{A} , a class of all free \mathbb{Z} -modules. It is well known that F is a quasi-continuous module and F is not a continuous module. Thus, F is an \mathcal{A} -C3 module and satisfies the property: there exists a submodule $N \in \mathcal{A}$ of F that is isomorphic to a direct summand of F is not a direct summand.

Proposition 2.15. *Let M be a right R -module and \mathcal{A} , a class of right R -modules and closed under isomorphisms and summands. If every factor module of M is \mathcal{A} -projective, then the following conditions are equivalent:*

- (1) *For any two direct summands M_1, M_2 of M such that $M_1, M_2 \in \mathcal{A}$, $M_1 + M_2$ is a direct summand of M .*
- (2) *M is an \mathcal{A} -C3 module.*
- (3) *For any decomposition $M = A_1 \oplus A_2$ with $A_1 \in \mathcal{A}$, then every homomorphism $f : A_1 \rightarrow A_2$ has the image a direct summand of A_2 .*
- (4) *Every submodule $N \in \mathcal{A}$ of M that is isomorphic to a direct summand of M is itself a direct summand.*
- (5) *Whenever X_1, X_2, \dots, X_n are direct summands of M and $X_1, X_2, \dots, X_n \in \mathcal{A}$, then $\sum_{i=1}^n X_i$ is a direct summand of M .*

Proof. (1) \Rightarrow (2) is obvious.

(2) \Rightarrow (3) \Rightarrow (1) are proved similarly to the argument proof of Proposition 2.12.

(4) \Rightarrow (2) is obvious.

(3) \Rightarrow (4). Let $\sigma : A \rightarrow B$ be an isomorphism with $A \in \mathcal{A}$ a summand of M and $B \leq M$. We need to show that B is a direct summand of M . Write $M = A \oplus T$ for

some submodule T of M . We have $A/A \cap B$ is an image of M and obtain that $A \cap B$ is a direct summand of A . Take $A = (A \cap B) \oplus C$ for some submodule C of A . Now $M = (A \cap B) \oplus (C \oplus T)$. Clearly, $A \cap [(C \oplus T) \cap B] = 0$ and $B = (A \cap B) \oplus [(C \oplus T) \cap B]$. Let $H := \sigma^{-1}((C \oplus T) \cap B)$. Then H is a submodule of A , $H \cap [(C \oplus T) \cap B] = 0$ and $A = H \oplus H'$ for some submodule H' of H . Note that $M = H \oplus (H' \oplus T)$. Consider the projection $\pi : M \rightarrow H' \oplus T$. Then

$$H \oplus [(C \oplus T) \cap B] = H \oplus \pi((C \oplus T) \cap B).$$

By (3), the image of the homomorphism $\pi|_{(C \oplus T) \cap B} \circ \sigma|_H : H \rightarrow H' \oplus T$ is a direct summand of $H' \oplus T$ since H is contained in \mathcal{A} . Write $H' \oplus T = \pi|_{(C \oplus T) \cap B} \sigma(H) \oplus K$ for some submodule K of $H' \oplus T$. Then $H' \oplus T = \pi((C \oplus T) \cap B) \oplus K$. It follows that

$$M = H \oplus \pi((C \oplus T) \cap B) \oplus K = H \oplus [(C \oplus T) \cap B] \oplus K.$$

By the modular law, $C \oplus T = [(C \oplus T) \cap B] \oplus [(H \oplus K) \cap (C \oplus T)]$. Thus

$$\begin{aligned} M &= (A \cap B) \oplus [(C \oplus T) \cap B] \oplus [(H \oplus K) \cap (C \oplus T)] \\ &= B \oplus [(H \oplus K) \cap (C \oplus T)]. \end{aligned}$$

The implication (1) \Rightarrow (5) is proved similarly to the argument proof of Proposition 2.13. \square

Corollary 2.16. *The following conditions are equivalent for a module M :*

- (1) *For any semisimple submodules A, B of M with $A \cong B \subset_d M$, $A \subset_d M$.*
- (2) *For any semisimple summands A, B of M , $A + B \subset_d M$.*
- (3) *For any semisimple summands A, B of M with $A \cap B = 0$, $A + B \subset_d M$.*
- (4) *Whenever X_1, \dots, X_n are semisimple summands of M and $X_1, \dots, X_n \in \mathcal{A}$, then $\sum_{i=1}^n X_i$ is a direct summand of M .*

Corollary 2.17. *Let Q be a quasi-injective module. If X_1, \dots, X_n are semisimple summands of Q , then $\sum_{i=1}^n X_i$ is a direct summand of Q .*

Corollary 2.18 ([6, Proposition 2.1]). *The following conditions are equivalent for a module M :*

- (1) *For any simple submodules A, B of M with $A \cong B \subset_d M$, $A \subset_d M$.*
- (2) *For any simple summands A, B of M with $A \cap B = 0$, $A \oplus B \subset_d M$.*
- (3) *For any finitely generated semisimple submodules A, B of M with $A \cong B \subset_d M$, $A \subset_d M$.*
- (4) *For any finitely generated semisimple summands A, B of M with $A \cap B = 0$, $A \oplus B \subset_d M$.*

3. CHARACTERIZATIONS OF RINGS

Lemma 3.1. *Let \mathcal{A} be a class of right R -modules with local endomorphisms and closed under isomorphisms. Assume that K and M are indecomposable right R -modules and not contained in \mathcal{A} . Then*

- (1) $N = M \oplus P$ is an \mathcal{A} -D3 module for all projective modules P .
- (2) $N = M \oplus E$ is an \mathcal{A} -C3 module for all injective modules E .
- (3) $N = M \oplus K$ is an \mathcal{A} -D3 module and an \mathcal{A} -C3 module.

Proof. (1) Let $N/A \cong S \subset_d N$ with $S \in \mathcal{A}$. By [5, Lemma 26.4], there exist a direct summand M_1 of M and a direct summand P_1 of P such that $N = S \oplus M_1 \oplus P_1$. Write $P = P_1 \oplus P_2$ for some submodule P_2 of P . Since M is an indecomposable module, we have either $M_1 = 0$ or $M = M_1$. If $M_1 = 0$, then $N = S \oplus P_1 = (M \oplus P_2) \oplus P_1$ and it follows that $M \oplus P_2 \cong S$, and hence $M \in \mathcal{A}$ contradicting. So $M_1 = M$. Then $N = S \oplus (M \oplus P_1) = (M \oplus P_1) \oplus P_2$. This gives $S \cong P_2$, and consequently $N/A \cong S$ is projective. Hence, A is a direct summand of N and (1) holds.

(2) Suppose that A is a submodule of N such that $A \simeq S$ with S a submodule of N and $S \in \mathcal{A}$. As in (1), we see that $N = S \oplus M_1 \oplus E_1$ with $M = M_1 \oplus M_2$ and $E = E_1 \oplus E_2$. Also, as in (1), $M_1 = M$. Therefore,

$$N = S \oplus M \oplus E_1 = M \oplus E = (M \oplus E_1) \oplus E_2.$$

It follows that $S \simeq E_2$ is an injective module. Thus A is a direct summand of N .

(3) We show that N has no a nonzero direct summand S with $S \in \mathcal{A}$. Assume on the contrary that there exists a non-zero summand $S \subset_d N$ with $S \in \mathcal{A}$. As, in (1), $N = S \oplus M_1 \oplus K_1$ with $M = M_1 \oplus M_2$ and $K = K_1 \oplus K_2$. Also, as in (1), $M_1 = M$. Therefore,

$$N = S \oplus M \oplus K_1 = M \oplus K.$$

Since K is indecomposable, $K = K_1$ or $K = K_2$. If $K = K_1$, then $S \oplus M \oplus K = M \oplus K$ and consequently $S = 0$, a contradiction. If $K = K_2$, then $K_1 = 0$ and so $S \oplus M = M \oplus K$. Therefore, $K \cong S$ and hence $K \in \mathcal{A}$, a contradiction. \square

Recall that a module is *uniserial* if the lattice of its submodules is totally ordered under inclusion. A ring R is called right *uniserial* if R_R is a uniserial module. A ring R is called *serial* if both modules ${}_R R$ and R_R are direct sums of uniserial modules.

Theorem 3.2. *Let R be a right artinian ring and \mathcal{A} , a class of right R -modules with local endomorphisms, containing all right simple right R -modules and closed under isomorphisms. If all right R -modules are \mathcal{A} -injective, then the following conditions are equivalent for a ring R :*

- (1) R is a serial artinian ring with $J^2(R) = 0$.
- (2) Every \mathcal{A} -C3 module is quasi-injective.
- (3) Every \mathcal{A} -C3 module is C3.

Proof. (1) \Rightarrow (2) Assume that R is an artinian serial ring with $J^2(R) = 0$. Then every right R -module is a direct sum of a semisimple module and an injective module. Furthermore, every injective module is a direct sum of cyclic uniserial modules. Let M be an \mathcal{A} -C3 module. We can write $M = (\oplus_{\mathcal{I}} S_i) \oplus (\oplus_{\mathcal{J}} E_j)$ where each S_i is simple if $i \in \mathcal{I}$ and $\oplus_{\mathcal{J}} E_j$ is injective where each E_j is cyclic uniserial non-simple if $j \in \mathcal{J}$. Note

that any E_j has length at 2 by [7, 13.3]. We show that M is a quasi-injective module. To show that M is quasi-injective, by [16, Proposition 1.17] it suffices to show that $\oplus_{\mathcal{I}} S_i$ is $\oplus_{\mathcal{J}} E_j$ -injective. By [16, Theorem 1.7], $\oplus_{\mathcal{I}} S_i$ is $\oplus_{\mathcal{J}} E_j$ -injective if and only if S_i is $\oplus_{\mathcal{J}} E_j$ -injective for all $i \in \mathcal{I}$. Furthermore, for any $i \in \mathcal{I}$, if S_i is E_j -injective for all $j \in \mathcal{J}$, then S_i is $\oplus_{\mathcal{J}} E_j$ -injective by [16, Proposition 1.5]. So, it suffices to show that S_i is E_j -injective for each $i \in \mathcal{I}$ and $j \in \mathcal{J}$. Suppose that E_j has a series $0 \subset X \subset E_j$. Let $f : A \rightarrow S_i$ be a homomorphism with $A \leq E_j$. If $A = 0$ or $A = E_j$ then it is obvious that f is extended to a homomorphism from E_j to S_i . Assume that $A = X$. If f is non-zero, then $X \simeq S_i$. As M is an \mathcal{A} -C3 module, X is a direct summand of M . It follows that $X = E_j$, a contradiction. Hence S_i is E_j -injective and so M is quasi-injective.

(2) \Rightarrow (3) This is clear.

(3) \Rightarrow (1) Let M be an indecomposable module. If $M \in \mathcal{A}$, then it is quasi-injective. Now, suppose that $M \notin \mathcal{A}$ and let $\iota : M \rightarrow E(M)$ be the inclusion. Then, by Lemma 3.1, $M \oplus E(M)$ is \mathcal{A} -C3 and by assumption, $M \oplus E(M)$ is a C3-module. It follows that $\text{Im}(\iota)$ is a direct summand of $E(M)$ by [4, Proposition 2.3]. Hence M is injective. Inasmuch as every indecomposable right R -module is quasi-injective, we infer from [9, Theorem 5.3] that R is an artinian serial ring. By [8, Theorem 25.4.2], every right R -module is a direct sum of uniserial modules. Now, by [7, 13.3], we only need to show that each uniserial module, say M , has length at most 2. Suppose that M has a series $0 \subset X \subset Y \subset M$ of length 3. Assume that $Y \in \mathcal{A}$. Then X is Y -injective and hence X is a direct summand of Y , a contradiction. It follows that $Y \notin \mathcal{A}$. By Lemma 3.1, $M \oplus Y$ is an \mathcal{A} -C3 module and then, by hypothesis, is a C3-module. Consequently, the natural inclusion, $\eta : Y \rightarrow M$ splits; i.e. $Y \subset_d M$ and so $Y = M$, a contradiction. Hence, R is an artinian ring with $J^2(R) = 0$. \square

Theorem 3.3. *Let R be a right artinian ring and \mathcal{A} , a class of right R -modules with local endomorphisms, containing all right simple right R -modules and closed under isomorphisms. If all right R -modules are \mathcal{A} -projective, then the following conditions are equivalent for a ring R :*

- (1) *R is a serial artinian ring with $J^2(R) = 0$.*
- (2) *Every \mathcal{A} -D3 module is quasi-projective.*
- (3) *Every \mathcal{A} -D3 module is D3.*

Proof. By Lemma 3.1 and [13, Theorem 4.4]. \square

Proposition 3.4. *Let \mathcal{A} be a class of right R -modules and closed under isomorphisms and summands. Then the following conditions are equivalent:*

- (1) *All modules $A \in \mathcal{A}$ are injective.*
- (2) *Every right R -module is \mathcal{A} -C3.*

Proof. (1) \Rightarrow (2) is obvious.

(2) \Rightarrow (1). Suppose that $A \in \mathcal{A}$. Then by (2), $A \oplus E(A)$ is an \mathcal{A} -C3 module. Call $\iota : A \rightarrow E(A)$ the inclusion map. By Proposition 2.11, $\text{Im}(\iota) = A$ is a direct summand of $E(A)$. Thus $A = E(A)$ is an injective module. \square

Corollary 3.5 ([6]). *The following conditions are equivalent for a ring R :*

- (1) R is a right V -ring.
- (2) Every right R -module is simple-direct-injective.

Proposition 3.6. *Let \mathcal{A} be a class of right R -modules and closed under isomorphisms and summands. Then the following conditions are equivalent:*

- (1) All modules $A \in \mathcal{A}$ are projective.
- (2) Every right R -module is \mathcal{A} -D3.

Proof. (1) \Rightarrow (2). Assume that M is a right R -module. Let M_1, M_2 be submodules of M with $M/M_1, M/M_2 \in \mathcal{A}$ and $M = M_1 + M_2$. It follows that $M/M_1, M/M_2$ are projective modules and the following isomorphism

$$M/(M_1 \cap M_2) = (M_1 + M_2)/(M_1 \cap M_2) \simeq M/M_1 \times M/M_2.$$

Then $M/(M_1 \cap M_2)$ is a projective module. We deduce that $M_1 \cap M_2$ is a direct summand of M . It shown that M is an \mathcal{A} -D3 module.

(2) \Rightarrow (1). Suppose that $A \in \mathcal{A}$. Call $\varphi : R^{(I)} \rightarrow A$ an epimorphism. Then $R^{(I)} \oplus A$ is an \mathcal{A} -D3 module. By Proposition 2.6, A is isomorphic to a direct summand of $R^{(I)}$. Thus A is a projective module. \square

Corollary 3.7 ([13]). *The following conditions are equivalent for a ring R :*

- (1) R is a semisimple artinian ring.
- (2) Every right R -module is simple-direct-projective.

Let M be a right R -module. M is called *regular* if every cyclic submodule of M is a direct summand. A right R -module is called *M -cyclic* if it is isomorphic to a factor module of M .

Lemma 3.8. *Let F be a regular module. Assume that $A \neq 0$ is a small finitely generated submodule of the factor module F/F_0 for some submodule F_0 of F and \mathcal{A} the class of all modules isomorphism to A . Then there exists a F -cyclic module M and satisfies the property: there is a submodule $N \in \mathcal{A}$ of M that is isomorphic to a direct summand of M and not a direct summand.*

Proof. By the hypothesis we have $((x_1R + x_2R + \cdots + x_mR) + F_0)/F_0 = A$ for some x_1, x_2, \dots, x_m of F . Since F is a regular module, $x_1R + x_2R + \cdots + x_mR = \pi(F)$, where $\pi \in \text{End}(F)$ and $\pi^2 = \pi$. Since A is a small submodule of F/F_0 , we have $F/F_0 = ((1 - \pi)F + F_0)/F_0$. It follows that there exist epimorphisms $f_1 : \pi(F) \rightarrow A$, $f_2 : (1 - \pi)(F) \rightarrow F/F_0$. It is easy to check $A \oplus (F/F_0)$ is an F -cyclic module. Call $M = A \oplus (F/F_0)$. Thus, the module $N := 0 \oplus A \simeq A$ is not a direct summand of M and isomorphic to a direct summand of M . \square

A module M is called a V -module if every simple module in $\sigma[M]$ is M -injective (see [19]). R is called a right V -ring if the right module R_R is a V -module.

Theorem 3.9. *The following conditions are equivalent for a regular module F :*

- (1) F is a V -module.
- (2) Every F -cyclic module M is an \mathcal{A} -C3 module where \mathcal{A} is the class of all simple submodules of M .

Proof. The implication (1) \Rightarrow (2) is obvious.

(2) \Rightarrow (1). Let $S \in \sigma[F]$ is a simple module and $E_F(S)$ is the injective hull of S in the category $\sigma[F]$. Assume that $E_F(S) \neq S$. As $E_F(S)$ is generated by F , there exists a homomorphism $f : F \rightarrow E_F(S)$ such that $f(F) \neq S$. Then S is a small submodule of $f(F) \simeq F/\text{Ker}(f)$. Call \mathcal{A} the class of all modules isomorphism to S . By Lemma 3.8, there exists a F -cyclic module M and satisfies the property: there is a submodule $N \in \mathcal{A}$ of M that is isomorphic to a direct summand of M and not a direct summand. We infer from Proposition 2.15 that M is not an \mathcal{A} -C3 module. This contradicts the condition of (2). \square

Corollary 3.10 ([6, Theorem 4.4.]). *A regular ring R is a right V -ring if and only if every cyclic right R -module is simple-direct-injective.*

REFERENCES

1. Abyzov, A. N. and Tuganbaev A. A. *Modules in which sums or intersections of two direct summands are direct summands*, Fundam. Prikl. Mat. **19**, 3-11, 2014.
2. Abyzov A. N. and Nhan T. H. N. *CS-Rickart Modules*, Lobachevskii Journal of Mathematics, **35**, 317-326, 2014.
3. Alkan, M. and Harmanci, A. *On Summand Sum and Summand Intersection Property of Modules*, Turkish J. Math, **26**, 131-147, 2002.
4. Amin, I. Ibrahim, Y. and Yousif, M. F. *C3-modules*, Algebra Colloq. **22**, 655-670, 2015.
5. Anderson, F. W. and Fuller, K. R. *Rings and Categories of Modules*, Springer-Verlag, New York, 1974.
6. Camillo, V. Ibrahim, Y. Yousif, M. and Zhou, Y. *Simple-direct-injective modules*, J. Algebra **420**, 39-53, 2014.
7. Dung, N. V. Huynh, D. V. Smith, P. F. and Wisbauer, R. *Extending modules*, Pitman Research Notes in Math. **313**, Longman, Harlow, New York, 1994.
8. Faith, C. *Algebra II. Ring Theory*, Springer-Verlag, New York, 1967.
9. Fuller, K. R. *On indecomposable injectives over artinian rings*, Pacific J. Math **29**, 115-135, 1968.
10. Garcia, J. L. *Properties of Direct Summands of Modules*, Comm. Algebra, **17**, 73-92, 1989.
11. Hamdouni, A. Harmanci, A. and Ç. Özcan, A. *Characterization of modules and rings by the summand intersection property and the summand sum property*, JP Jour.Algebra, Number Theory & Appl. **5**, 469-490, 2005.
12. Hausen, J. *Modules with the Summand Intersection Property*, Comm. Algebra **17**, 135-148, 1989.
13. Ibrahim, Y. Kossan, M. T. Quynh, T.C. and Yousif, M. *Simple-direct-projective modules*, to appear in Comm. Algebra, 2015.
14. Kaplansky, I. *Infinite Abelian Groups*, Univ. of Michigan Press, Ann Arbor, 1969.

15. Keskin Tütüncü, D. Mohamed, S.H. and Orhan Ertas, N. *Mixed injective modules*, Glasg. Math. J. **52**, 111-120, 2010.
16. Mohammed, S. H. and Müller, B. J. *Continuous and Discrete Modules*, London Math. Soc. LN **147**: Cambridge Univ. Press., 1990.
17. Quynh, T. C. Kosan, M. T. and Thuyet, L. V. *On (semi)regular morphisms*, Comm. Algebra **41**, 2933-2947, 2013.
18. Wilson, G. V. *Modules with the Direct Summand Intersection Property*, Comm. Algebra **14**, 21-38, 1986.
19. Wisbauer, R. *Foundations of Module and Ring Theory*, Gordon and Breach. Reading, 1991.
20. Yousif, M.F. Amin, I. and Ibrahim, Y. *D3-modules*. Commun. Algebra **42**, 578-592, 2014.

DEPARTMENT OF ALGEBRA AND MATHEMATICAL LOGIC, KAZAN (VOLGA REGION) FEDERAL UNIVERSITY, 18 KREMLYOVSKAYA STR., KAZAN, 420008 RUSSIA
E-mail address: aabyzov@ksu.ru, Adel.Abyzov@ksu.ru

DEPARTMENT OF MATHEMATICS, DANANG UNIVERSITY, 459 TON DUC THANG, DANANG CITY, VIETNAM
E-mail address: tcquynh@dce.udn.vn; tcquynh@live.com

DEPARTMENT OF ALGEBRA AND MATHEMATICAL LOGIC, KAZAN (VOLGA REGION) FEDERAL UNIVERSITY, 18 KREMLYOVSKAYA STR., KAZAN, 420008 RUSSIA
E-mail address: tranhoaingocnhan@gmail.com